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Complete electroweak matching for radiative B decays

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Abstract: We compute the complete two-loop $\mathrm{O}(\alpha)$ Wilson coefficients relevant for radiative decays of the B meson in the SM. This is a necessary step in the calculation of the $\mathrm{O}(\alpha\alpha_s^n \ln^n m_b/M_W)$ corrections and improves on our previous analysis of electroweak effects in $B \to X_s\gamma$. We describe in detail several interesting technical aspects of the calculation and include all dominant QED matrix elements. In our final result, we neglect only terms originated from the unknown $\mathrm{O}(\alpha\alpha_s)$ evolution of the Wilson coefficients and some suppressed two-loop matrix elements. Due to the compensation among different effects, we find that non-trivial electroweak corrections decrease the branching ratio by about 3.6% for a light Higgs boson, very close to our previous result. The corresponding SM prediction for the branching ratio with $E_\gamma > 1.6\,\mathrm{GeV}$ is $(3.61 \pm 0.30) \times 10^{-4}$.

Keywords: Standard Model, Rare Decays, B-Physics.

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Contents

1. Introduction

The Standard Model (SM) calculation of the branching ratio for the inclusive radiative decay $B \to X_s\gamma$ — BR_γ in the following — has reached a high degree of sophistication (see [1] for a complete list of references and an up-to-date analysis). Besides Leading Logarithmic $\mathrm{O}(\alpha_s^n L^n)$ ($L = \ln m_b/M_W$) and Next-to-Leading Logarithmic (NLO) $\mathrm{O}(\alpha_s^n L^{n-1})$ QCD corrections and non-perturbative Heavy Quark Effective Theory contributions, electroweak effects are known to play a non-negligible role [2]–[6]. In a previous work [2], we have considered in detail the electroweak corrections to this process, devoting special attention to the interplay between QCD and electroweak effects. Photonic interactions generate logarithmically enhanced contributions which are suppressed by a factor α/α_s with respect to the QCD ones. The leading QED effects are therefore $\mathrm{O}(\alpha\alpha_s^{n-1}L^n)$ and are known completely [6], while genuine electroweak corrections involving Z^0 and W bosons start at the next order in the resummed logarithmic expansion. Ideally, one would like to have all these $\mathrm{O}(\alpha\alpha_s^n L^n)$ corrections under control. Since the $\mathrm{O}(\alpha)$ contributions to the coefficients of the four quark operators are all known [2, 7], this would entail the following steps [2, 8]:

(i) the calculation of the two-loop $\mathrm{O}(\alpha)$ matching conditions for the magnetic operators Q_7^γ and Q_8^g at a scale $\mathrm{O}(M_W)$;

(ii) the QED-QCD evolution of the Wilson coefficients down to the B mass scale, including the calculation of the two and three-loop $\mathrm{O}(\alpha\alpha_s)$ anomalous dimension matrix;

(iii) the calculation of the one-loop and two-loop QED matrix elements of the various operators as well as of some yet unknown two-loop QCD matrix elements.

Our analysis in [2] was based on the simplifying assumption that terms vanishing as $s_W \equiv \sin\theta_W \to 0$ can be neglected, unless they are enhanced by powers of the top mass M_t. In this case, introducing the $SU(2)_L$ coupling g and $\alpha_W = g^2/4\pi$, all electroweak corrections are in fact $O(\alpha_W \alpha_s^n L^n)$ or $O(\alpha M_t^2/M_W^2 \alpha_s^n L^n)$ and are included by step (i) only. Moreover, the calculation of the Wilson coefficients simplifies considerably. Although reasonable, this assumption should be verified — keep in mind that $s_W^2 \approx 0.23$. In particular, Z^0 boson corrections to the one-loop $b \to s\gamma$ magnetic penguin diagrams give rise to $O(s_W^2)$ terms which are not formally suppressed by an electric charge factor $Q_d^2 = 1/9$ or $Q_u|Q_d| = 2/9$, unlike the purely QED corrections of steps (ii) and (iii). This happens, for instance, because of the mass difference between Z^0 and W bosons. Such $O(s_W^2)$ terms originate at the electroweak scale and affect only step (i).

In this note we extend our calculation [2] and compute the full $O(\alpha)$ contribution to the Wilson coefficients of the $b \to s$ magnetic operators, thus completing step (i). The main difference (and technical hurdle) with respect to [2] is due to the presence of virtual photons in the two-loop SM diagrams. The resulting infrared (IR) divergences are removed in the matching with the effective low-energy theory of quarks, photons and gluons. Several subtleties arise in the calculation, mostly linked to the presence of unphysical operators. This is explained in detail in section 2, while section 3 deals with the QED-QCD evolution of the coefficients and illustrates how $O(\alpha\alpha_s^n L^n)$ effects should be taken into account in the calculation of BR_γ. We also include all dominant $O(\alpha)$ matrix elements and conclude reconsidering the SM prediction of BR_γ.

2. The $O(\alpha)$ matching

Let us briefly recall the formalism. We work in the framework of an effective low-energy theory with five active quarks, photons and gluons, obtained by integrating out heavy degrees of freedom characterized by a mass scale $M \geq M_W$. In the leading order of the operator product expansion the effective off-shell hamiltonian relevant for the $b \to s\gamma$ and $b \to sg$ transition at a scale μ is given by

$$\mathcal{H}_{\rm eff} = -\frac{G_F}{\sqrt{2}} V_{ts}^* V_{tb} \left[\sum_{i=1}^{16} C_i(\mu) Q_i + C_7^\gamma(\mu) Q_7^\gamma + C_8^g(\mu) Q_8^g \right]. \tag{2.1}$$

Here V_{ij} are the CKM matrix elements and $C_i(\mu)$, $C_7^\gamma(\mu)$ and $C_8^g(\mu)$ denote the Wilson coefficients of the following set of gauge invariant operators [6, 9, 10, 11]

$$\begin{aligned}
Q_1 &= (\bar{s}_L \gamma_\mu T^a c_L)(\bar{c}_L \gamma^\mu T^a b_L)\,, & Q_2 &= (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L)\,, \\
Q_3 &= (\bar{s}_L \gamma_\mu b_L) \textstyle\sum_q (\bar{q}\gamma^\mu q)\,, & Q_4 &= (\bar{s}_L \gamma_\mu T^a b_L) \textstyle\sum_q (\bar{q}\gamma^\mu T^a q)\,, \\
Q_5 &= (\bar{s}_L \gamma_\mu \gamma_\nu \gamma_\rho b_L) \textstyle\sum_q (\bar{q}\gamma^\mu \gamma^\nu \gamma^\rho q)\,, & Q_6 &= (\bar{s}_L \gamma_\mu \gamma_\nu \gamma_\rho T^a b_L) \textstyle\sum_q (\bar{q}\gamma^\mu \gamma^\nu \gamma^\rho T^a q)\,, \\
Q_7 &= (\bar{s}_L \gamma_\mu b_L) \textstyle\sum_q Q_q (\bar{q}\gamma^\mu q)\,, & Q_8 &= (\bar{s}_L \gamma_\mu T^a b_L) \textstyle\sum_q Q_q (\bar{q}\gamma^\mu T^a q)\,,
\end{aligned}$$

$$
\begin{aligned}
Q_9 &= (\bar{s}_L\gamma_\mu\gamma_\nu\gamma_\rho b_L)\textstyle\sum_q Q_q(\bar{q}\gamma^\mu\gamma^\nu\gamma^\rho q)\,, & Q_{10} &= (\bar{s}_L\gamma_\mu\gamma_\nu\gamma_\rho T^a b_L)\textstyle\sum_q Q_q(\bar{q}\gamma^\mu\gamma^\nu\gamma^\rho T^a q)\,,\\
Q_7^\gamma &= \frac{e}{16\pi^2} m_b(\bar{s}_L\sigma^{\mu\nu}b_R)F_{\mu\nu}\,, & Q_8^g &= \frac{g_s}{16\pi^2} m_b(\bar{s}_L\sigma^{\mu\nu}T^a b_R)G^a_{\mu\nu}\,,\\
Q_{11} &= \frac{1}{e}\bar{s}_L\gamma^\mu b_L\partial^\nu F_{\mu\nu} + Q_7\,, & Q_{12} &= \frac{1}{g_s}\bar{s}_L\gamma^\mu T^a b_L D^\nu G^a_{\mu\nu} + Q_4\,,
\end{aligned}
$$

$$
Q_{13} = \frac{ie}{16\pi^2}\left[\bar{s}_L \overleftarrow{\not{D}}\, \sigma^{\mu\nu} b_L F_{\mu\nu} - F_{\mu\nu}\bar{s}_L\sigma^{\mu\nu}\not{D} b_L\right] + Q_7^\gamma\,, \qquad Q_{15} = \frac{1}{16\pi^2} m_b \bar{s}_L \not{D}\not{D} b_R\,,
$$

$$
Q_{14} = \frac{ig_s}{16\pi^2}\left[\bar{s}_L \overleftarrow{\not{D}}\, \sigma^{\mu\nu} T^a b_L G^a_{\mu\nu} - G^a_{\mu\nu}\bar{s}_L T^a\sigma^{\mu\nu}\not{D} b_L\right] + Q_8^g\,,\; Q_{16} = \frac{i}{16\pi^2}\bar{s}_L\not{D}\not{D}\not{D} b_L\,, \tag{2.2}
$$

where e (g_s) is the electromagnetic (strong) coupling constant, $q_{L,R}$ are the chiral quark fields, $F_{\mu\nu}$ ($G^a_{\mu\nu}$) is the electromagnetic (gluonic) field strength tensor, D_μ is the covariant derivative of the gauge group $\mathrm{SU}(3)_C\times\mathrm{U}(1)_Q$ and T^a are the colour matrices, normalized so that $\mathrm{Tr}(T^aT^b) = \delta^{ab}/2$. The s-quark mass is neglected in eq. (2.2) and in the following. Notice that at the order we are going to work it is not necessary to consider the analogues of Q_1 and Q_2 involving the u-quark instead of the c-quark.

The above set of operators closes off-shell under QCD and QED renormalization, up to non-physical (evanescent) operators that vanish in four dimensions [9, 11]. It consists of the current-current operators Q_1–Q_2, the QCD penguin operators Q_3–Q_6, the electroweak penguin operators Q_7–Q_{10} and the magnetic moment type operators Q_7^γ and Q_8^g. It is the QED renormalization that forces us to introduce the operators Q_7–Q_{10}, in which the sum of the quark flavors is weighted by the electric charges Q_q. The remaining six operators Q_{11}–Q_{16}, characteristic of the process $b\to s\gamma\,(g)$, were chosen in such a way that they vanish on-shell up to total derivatives. Only operators of dimension five or six are retained. Higher dimension operators are suppressed by at least one power of m_b^2/M_W^2, while those of lower dimensionality can be removed by choosing suitable renormalization conditions in the full theory [12]. In the present case this is achieved by requiring that all flavor off-diagonal quark two-point functions which appear at the one-loop level in the full theory vanish when the equations of motion (EOM) are applied, i.e. by using LSZ on-shell conditions on the external quark lines. We also use these renormalization conditions for internal virtual quarks in the full SM calculation and implement in this way a gauge invariant $\mathrm{O}(\alpha)$ definition of the CKM matrix [13].

In order to obtain the Wilson coefficients of the magnetic operator Q_7^γ, we calculate the off-shell amplitude $b\to s\gamma$ in the full SM and in the effective theory at $\mathrm{O}(\alpha)$ and match the two results. Retaining only the leading terms in $1/M_W^2$ the off-shell amplitude in the full theory can be written in the following form[1]

$$
\mathcal{A}_{\rm full} = -\frac{G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_i A_i \left\langle s\gamma\left|Q_i\right|b\right\rangle^{(0)}\,, \tag{2.3}
$$

[1] In eqs. (2.3) and (2.8), the sum runs over Q_1–Q_{16}, Q_7^γ, Q_8^g and, as we will explain later on, some evanescent operators.

where $\langle s\gamma|Q_i|b\rangle^{(0)}$ are the tree-level matrix elements of the operators in eq. (2.2). The perturbative expansion of the coefficients A_i reads

$$A_i = A_i^{(0)} + \frac{\alpha}{4\pi} A_{i,e}^{(1)} \,. \tag{2.4}$$

We calculate analytically the relevant one and two-loop amplitudes starting from the diagrams generated by *FeynArts 2.2* [14] and retaining only terms which project on Q_7^γ after use of the EOM. All ultraviolet (UV) divergences in $A_{7,e}^{\gamma(1)}$ are removed by electroweak renormalization. We follow closely the procedure outlined in [2, 15]. The only additional ingredients not explicitly given in those papers are the right-handed down quark wave function renormalization ($c_W^2 = 1 - s_W^2$)

$$\delta(Z_d^R)_{ij} = \frac{g^2}{16\pi^2}\frac{Q_d^2 s_W^4}{c_W^2}\delta_{ij}\left[\frac{1}{\epsilon} - \frac{1}{2} - \ln\frac{M_Z^2}{\mu^2}\right], \tag{2.5}$$

and the complete b-quark on-shell mass counterterm ($x_t = \overline{M}_t^2/M_W^2$)

$$\begin{aligned}\frac{\delta m_b}{m_b} = \frac{g^2}{16\pi^2}\left(\frac{\bar{\mu}^2}{\overline{M}_t^2}\right)^{\epsilon}\Bigg[&\frac{1}{8\epsilon}(3x_t - 2) - \frac{2 + 9x_t - 5x_t^2}{16(x_t-1)} - \frac{2 - 7x_t + 2x_t^2}{8(x_t-1)^2}\ln x_t + \\ &+ \frac{1}{c_W^2}\Bigg(\frac{1}{16} - \frac{5}{12}s_W^2 + \frac{5}{18}s_W^4 + \\ &\qquad + \left(\frac{1}{8} + \frac{s_W^2}{2} - \frac{s_W^4}{3}\right)\left(\ln\frac{M_Z^2}{\overline{M}_t^2} - \frac{1}{\epsilon}\right)\Bigg)\Bigg],\end{aligned} \tag{2.6}$$

which had not been given in [2, 15]. Notice that eqs. (2.5) and (2.6) do *not* include the photon contribution, for a reason that will become clear in a moment. The top mass $\overline{M}_t$ is renormalized on-shell as far as electroweak effects are concerned, while we use an $\overline{\rm MS}$ definition at a scale μ for the QCD renormalization.

We work in the background field gauge (BFG) [16]. This reduces the number of diagrams to be considered. Moreover, if the electric charge is normalized at $q^2 = 0$, as it is natural to do [3], its counterterm cancels identically against the background photon wave function renormalization factor, due to the BFG Ward identity [17]. The same holds in the case of an external gluon in the $\overline{\rm MS}$ scheme. The regularization problems related to the definition of γ_5 in $n = 4 - 2\epsilon$ dimensions are avoided as described in [2] and we employ the naive dimensional regularization scheme with anticommuting γ_5 (NDR) throughout the paper.

For what concerns the regularization of the IR divergences, we have adopted two different methods and found identical results for the Wilson coefficients. In the first method the IR divergences are regulated by quark masses (see [10]), while the second method consists in using dimensional regularization for both UV and IR divergences [11].

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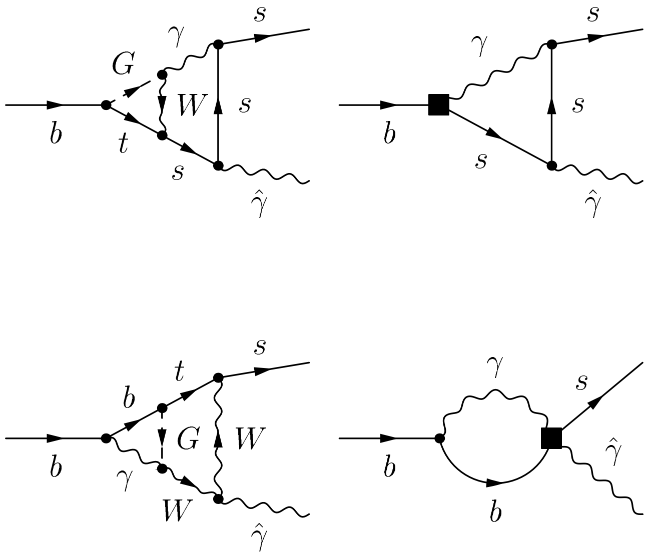

Figure 1: Subdiagrams involving the coupling of quantum photons γ to the W and its corresponding Goldstone boson (left-hand side) contribute to gauge variant operators (right-hand side), as explained in the text. $\hat{\gamma}$ denotes a background photon.

A second step involves the calculation of the off-shell amplitude in the QED effective theory. In general, we need effective vertices with both background and quantum photons. Interestingly, the latter introduce some gauge variant operators at $\mathrm{O}(\alpha)$. In fact, on the full theory side there are heavy particle subdiagrams (see figure 1) that are coupled to quantum photons and contribute to gauge variant operators not included in the operator basis of eq. (2.2). This is due to the R_ξ gauge coupling of quantum photons with W and Goldstone bosons and is different from what happens in the case of the off-shell $\mathrm{O}(\alpha_s)$ matching [10, 11]. Indeed, at $\mathrm{O}(\alpha_s)$ only quark-gluon couplings and trilinear quantum-quantum-background gluon couplings are relevant and no gauge variant operator is induced. The appearance of gauge variant operators in the SM amplitudes is not surprising [18] (see [19] for an example).

We have explicitly verified that it is not necessary to take any gauge variant operator into account on the effective theory side. This follows from well-known theorems on the renormalization of gauge invariant operators[2] [18, 21]: gauge variant operators that mix with gauge invariant operators can be chosen so that they are all BRST-exact, i.e. they can be written as the BRST-variation of some other operators, modulo terms vanishing by the EOM. Therefore, while gauge invariant operators

[2]The theorems apply to Yang-Mills theories, but extend to the full SM after imposing the anti-ghost equation [20].

generally mix into gauge variant operators,[3] the opposite is not true. Since we are eventually interested in the matrix elements of physical operators only, we do not need to include gauge variant effective operators in our basis. Of course this holds only as long as the regularization respects the symmetries, like it is in our case.

In a similar way and because of the same theorems, the operators that vanish by EOM in the basis (2.2) do not mix into the physical operators of the same basis, and the renormalization mixing matrix is block triangular. This property drastically simplifies the computation at hand, as we will see in a moment. In particular, the renormalization mixing matrix $\hat{Z}$ is such that $Z_{ij} = 0$ when Q_i is EOM-vanishing and Q_j is a physical operator.

We have seen that effective vertices involving quantum photons are induced in the calculation. Even though contributions to gauge variant operators turn out to be irrelevant, the distinction is important in the case that the calculation is performed using quark masses to regularize IR divergences. For example, it turns out that the gauge invariant part of the off-shell $b \to s\gamma$ effective vertex depends on whether the external photon is quantum or background. This can be explained by noting that the operators involving only background fields are combinations of truly gauge invariant operators and of operators containing also quantum gauge fields. This follows, e.g. from a decomposition of the kind $\hat{\not{D}} = \not{D} + ieQ_q\not{Q}$, where we used a hat to denote covariant background derivative, Q_μ for the quantum photon field, and Q_q for the electric charge. The operators containing quantum gauge bosons eventually decouple from the calculation, as they are not gauge invariant. Because of the above decomposition, the coefficients of the operators involving only background fields are related to the coefficients of the operators in (2.2), as can be seen using Slavnov-Taylor identities of the kind used in [22].

The effective theory calculation depends crucially on the IR regularization. The first method mentioned above (quark masses as regulator) can be applied at the diagrammatic level [10]. The effective theory diagrams are obtained by replacing hard (heavy mass) subdiagrams in the two-loop SM amplitudes with their Taylor expansions with respect to their external momenta. In principle, this method does not require a discussion of the effective operators. On the other hand, it is relatively complicated to implement. Here, we limit our discussion to the second method only, following [11]. In order to get the renormalized off-shell amplitude on the effective theory side, we need to reexpress eq. (2.1) in terms of renormalized quantities. The relations between the bare and the QED renormalized quantities are as follows

$$
\begin{aligned}
&e_0 = Z_e e\,, \qquad m_{b,0} = Z_{m_b} m_b\,, \qquad A_0^\mu = Z_e^{-1} A^\mu\,, \\
&q_0 = Z_q^{1/2} q\,, \qquad C_{i,0} = \sum_j C_j Z_{ji}\,,
\end{aligned}
\tag{2.7}
$$

[3]At $O(\alpha)$ the operators in our basis do *not* actually mix into gauge variant operators.

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with Z_e, Z_{m_b}, Z_q and Z_{ij} the renormalization constant of the charge, the b-quark mass, the quark fields and the Wilson coefficients, respectively. The relation between the renormalization of the gauge field and of the electric charge is a direct consequence of the QED Ward identity.

After renormalization the off-shell amplitude in the effective theory is given by

$$\mathcal{A}_{\rm eff} = -\frac{G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i,j} C_j Z_{ji} \widetilde{Z}_i \left\langle s\gamma \left| Q_i \right| b \right\rangle \,, \tag{2.8}$$

where $\widetilde{Z}_i$ denotes a product of Z_e, Z_m and Z_q depending on the particular structure of the operator Q_i and the Wilson coefficients may be expanded in powers of α as follows

$$C_i(\mu) = C_i^{(0)}(\mu) + \frac{\alpha}{4\pi} C_{i,e}^{(1)}(\mu) \,. \tag{2.9}$$

As long as we are only interested in the Wilson coefficient of the magnetic photon penguin operator, it is sufficient to keep only terms proportional to $\langle s\gamma|Q_7^\gamma|b\rangle$ in eq. (2.8). Using the short hand notation $\langle Q_7^\gamma\rangle \equiv \langle s\gamma|Q_7^\gamma|b\rangle$, the part of the off-shell amplitude in the effective theory needed for the matching of C_7^γ is then written as

$$\mathcal{A}_{\rm eff} \sim -\frac{G_F}{\sqrt{2}} V_{ts}^* V_{tb} \left[Z_q Z_{m_b} \sum_j C_j Z_{j,7\gamma} + Z_q (Z_{m_b} - 1) \sum_j C_j Z_{j,13} \right] \langle Q_7^\gamma \rangle \,, \tag{2.10}$$

where the second term proportional to $Z_{j,13}$ originates from the renormalization of the operator Q_{13}.

Notice that the QED quark field renormalization on the effective side can be avoided as it cancels in the matching against the photon contribution to the corresponding term in the SM. The same applies to the renormalization of the b-quark mass, which is retained only up to linear terms. Consequently, after checking the cancellation of the UV divergences in (2.4) we have omitted the photon contributions in eqs. (2.5) and (2.6) and simultaneously set Z_{m_b} and Z_q to unity in the effective theory. This simplifies the following considerations.

Adopting the $\overline{\rm MS}$ scheme for the operator renormalization the corresponding renormalization constants can be written as

$$Z_{ij} = \delta_{ij} + \frac{\alpha}{4\pi}\frac{1}{\epsilon} Z_{ij}^{(1)} + Z_{ij}^{(0)} \,. \tag{2.11}$$

The renormalization constants $Z_{ij}^{(1)}$ are found by calculating the UV divergent parts of Feynman diagrams in the effective theory. Within the scope of this computation, it is essential to carefully distinguish UV from IR singularities. As explained in ref. [23], this can be done most easily by introducing a common mass parameter into all the propagator denominators including the photon ones. All renormalization constants in the effective theory up to two loops are known from previous anomalous dimension calculations [5, 6]. As we shall see later on, only five entries of the anomalous dimension matrix are relevant in the present computation and we have recalculated these elements to check the results mentioned above. Our results are in full agreement with refs. [5, 6] and we will give the numerical values of the required renormalization constants below eqs. (2.13) and (2.16).

The last term in eq. (2.11) implies a finite renormalization of Q_i at zeroth order. Indeed, in situations where evanescent operators are present, the standard practice is to extend the $\overline{\mathrm{MS}}$ scheme and to allow for a finite operator renormalization. The finite terms $Z_{ij}^{(0)}$ differ from zero when Q_i is an evanescent operator and Q_j is not, and their values are fixed by requiring that renormalized matrix elements of evanescent operators vanish in $n = 4$ dimensions [24, 25]. This requirement also ensures that evanescent operators do not mix into physical ones [24]. Furthermore, in the case of the $b \to s\gamma$ calculation, it is well known [26] that some four quark operators can mix into the magnetic operators through one-loop diagrams at zeroth order in α and α_s. Thus, not only we have finite terms in eq. (2.11), but they appear at the lowest order in the coupling constant.

The computation of the necessary matrix elements on the effective side is trivial, as we can set all the light particles masses to zero.[4] Accordingly, all loop diagrams on the effective side vanish in dimensional regularization, because of the cancellation between UV and IR divergences. Therefore only the tree-level matrix elements $\langle Q_i \rangle^{(0)}$ are different from zero and higher order matrix elements do not play any role in the matching. Notice that due to the cancellation of UV and IR singularities the UV counterterms present in the tree-level matrix elements reproduce precisely the IR divergences in the effective theory. Furthermore, the IR divergence on the effective side has to be equal to the IR singularity on the SM side, to guarantee that the final results of the Wilson coefficients are free of IR poles. Eventually, all $1/\epsilon$ poles cancel out in C_7^γ, if the full and the effective theory are matched in the correct way.

Bearing all this in mind, we are now able to extract from eq. (2.10) those terms which are actually needed to calculate the $\mathrm{O}(\alpha)$ correction to the Wilson coefficient of the magnetic operator. First of all, we have to perform the tree-level matching by computing the relevant diagrams for the various operator insertions. Only C_2, C_7^γ, C_8^g and C_{11}–C_{16} are found to be non vanishing at leading order. However, due to the triangularity of the mixing matrix the coefficients C_{11}–C_{16} do not contribute to the first term in eq. (2.10) and therefore will not affect the matching conditions at the next order. Furthermore, as we set Z_{m_b} equal to one also the term proportional to $(Z_{m_b} - 1)$ in eq. (2.10) does not contribute to C_7^γ at $\mathrm{O}(\alpha)$. Using $Z_q = 1$ we

[4]We include only terms that are linear in the b-quark mass. They originate from use of the EOM only.

thus obtain

$$\mathcal{A}_{\rm eff} \sim -\frac{G_F}{\sqrt{2}} V_{ts}^* V_{tb} \left(C_7^{\gamma(0)} + \frac{\alpha}{4\pi} \left[C_{7,e}^{\gamma(1)} + \frac{1}{\epsilon} \left(Z_{2,7\gamma}^{(1)} C_2^{(0)} + Z_{7\gamma,7\gamma}^{(1)} C_7^{\gamma(0)} \right) + \right.\right.$$
$$\left.\left. + \sum_i Z_{E_i,7\gamma}^{(0)} C_{i,e}^{E,(1)} \right] \right) \langle Q_7^\gamma \rangle^{(0)} \,. \qquad (2.12)$$

It is quite remarkable that, with the exception of the last term, only *physical* operators play a role in this expression, even though the calculation has been performed off-shell.

The matching procedure between the full and the effective theory establishes the initial conditions for the Wilson coefficients a scale $\mu_W = \mathrm{O}(M_W)$. Comparing eqs. (2.3), (2.4) and (2.12), the matching condition $\mathcal{A}_{\rm full}(\mu_W) = \mathcal{A}_{\rm eff}(\mu_W)$ translates into the following identities

$$\begin{aligned} C_7^{\gamma(0)}(\mu_W) &= A_7^{\gamma(0)}(\mu_W)\,, \\ C_{7,e}^{\gamma(1)}(\mu_W) &= A_{7,e}^{\gamma(1)}(\mu_W) - \frac{1}{\epsilon}\left(Z_{2,7\gamma}^{(1)} C_2^{(0)}(\mu_W) + Z_{7\gamma,7\gamma}^{(1)} C_7^{\gamma(0)}(\mu_W) \right) - \\ &\quad - \sum_i Z_{E_i,7\gamma}^{(0)} C_{i,e}^{E(1)}(\mu_W)\,, \end{aligned} \qquad (2.13)$$

from which the Wilson coefficient of the magnetic operator up to $\mathrm{O}(\alpha)$ can be calculated. The leading order initial condition for the Wilson coefficient of Q_2 is simply $C_2^{(0)}(\mu_W) = 1$ and the elements of the mixing matrix needed for the next leading order matching of C_7^γ are $Z_{2,7\gamma}^{(1)} = -58/243$ and $Z_{7\gamma,7\gamma}^{(1)} = 8/9$. Note that the renormalization constant $Z_{2,7\gamma}^{(1)}$, related to the mixing of the operators Q_2 and Q_7^γ, is obtained from a two-loop calculation, as opposed to $Z_{7\gamma,7\gamma}^{(1)}$ which only involves a one-loop calculation. Whereas $Z_{7\gamma,7\gamma}^{(1)}$ is regularization and renormalization scheme independent, $Z_{2,7\gamma}^{(1)}$ is scheme dependent. The value for $Z_{2,7\gamma}^{(1)}$ given above corresponds to the NDR scheme — see [7]. Notice also that in eq. (2.13) the $\mathrm{O}(\epsilon)$ terms of $C_7^{\gamma(0)}$ yield a finite contribution when combined with the $1/\epsilon$ pole proportional to $Z_{7\gamma,7\gamma}^{(1)}$. Indeed, the leading order matching needs to be performed up to $\mathrm{O}(\epsilon)$. Explicit formulas for the initial condition of $C_7^{\gamma(0)}$ including $\mathrm{O}(\epsilon)$ terms can be found in [27, 28].

For what concerns the last term in eq. (2.13), it is necessary to introduce the following evanescent operators

$$\begin{aligned} Q_1^E &= (\bar{s}_L \gamma_\mu b_L) \sum\nolimits_q (\bar{q}_L \gamma^\mu q_L) + (1 + a_1 \epsilon) \left(\frac{1}{3} Q_3 - \frac{1}{12} Q_5 \right), \\ Q_2^E &= (\bar{s}_L \gamma_\mu b_L) \sum\nolimits_q Q_q (\bar{q}_L \gamma^\mu q_L) + (1 + a_2 \epsilon) \left(\frac{1}{3} Q_7 - \frac{1}{12} Q_9 \right), \\ Q_3^E &= (\bar{s}_L \gamma_\mu b_L) \sum\nolimits_q Q_q (\bar{q}_R \gamma^\mu q_R) - (1 + a_3 \epsilon) \left(\frac{4}{3} Q_7 - \frac{1}{12} Q_9 \right), \end{aligned} \qquad (2.14)$$

where a_i are arbitrary constants. In NDR inserting these operators into the one-loop $b \to s\gamma$ penguin diagrams yields $Z_{E_1,7\gamma} = 4/9$, $Z_{E_2,7\gamma} = -4/27$ and $Z_{E_3,7\gamma} = 4/27$. Notice that the last term in eq. (2.13) does not depend on the special choice of evanescent operators adopted above, but it does depend on the choice of physical operators. For instance, in the operator basis of [7], all evanescent operators that project on Q_7^γ have vanishing Wilson coefficients, both at $\mathrm{O}(\alpha)$ and $\mathrm{O}(\alpha_s)$. Therefore, in this basis evanescent operators do not affect the matching equations and is it not necessary to introduce them in eq. (2.11). Curiously, in the operators basis of eq. (2.2) the same holds only at $\mathrm{O}(\alpha_s)$.

We have verified that all $1/\epsilon$ poles cancel in eq. (2.13), and that the result for $C_{7,e}^{\gamma(1)}$ coincides with the one obtained using quark masses for the IR regularization. In the latter case evanescent operators do not play any role in the matching, as their contribution to the matrix elements cancels against a corresponding term stemming from the finite renormalization $Z_{E_i,7\gamma}^{(0)}$.

Let us now turn to the matching for the Wilson coefficient of the chromomagnetic penguin operator Q_8^g. The calculation for the $b \to s$ gluon off-shell amplitude proceeds in the same way as above. Adopting the notation $\langle Q_8^g \rangle \equiv \langle sg|Q_8^g|b\rangle$, we see that the analogue of eq. (2.12) is

$$\mathcal{A}_{\rm eff} \sim -\frac{G_F}{\sqrt{2}} V_{ts}^* V_{tb} \left(C_8^{g(0)} + \frac{\alpha}{4\pi} \left[C_{8,e}^{g(1)} + \frac{1}{\epsilon}\left(Z_{2,8g}^{(1)} C_2^{(0)} + Z_{7\gamma,8g}^{(1)} C_7^{\gamma(0)} + Z_{8g,8g}^{(1)} C_8^{g(0)} \right) + \right.\right.$$
$$\left.\left. + \sum_i Z_{E_i,8g}^{(0)} C_{i,e}^{E(1)} \right] \right) \langle Q_8^g \rangle^{(0)} \,, \qquad (2.15)$$

from which we obtain

$$\begin{aligned} C_8^{g(0)}(\mu_W) &= A_8^{g(0)}(\mu_W)\,, \\ C_{8,e}^{g(1)}(\mu_W) &= A_{8,e}^{g(1)}(\mu_W) - \frac{1}{\epsilon}\left(Z_{2,8g}^{(1)} C_2^{(0)}(\mu_W) + Z_{7\gamma,8g}^{(1)} C_7^{\gamma(0)}(\mu_W) + Z_{8g,8g}^{(1)} C_8^{g(0)}(\mu_W) \right) - \\ &\quad - \sum_i Z_{E_i,8g}^{(0)} C_{i,e}^{E(1)}(\mu_W)\,, \end{aligned} \qquad (2.16)$$

where $Z_{2,8g}^{(1)} = -23/81$, $Z_{7\gamma,8g}^{(1)} = -4/3$ and $Z_{8g,8g}^{(1)} = 4/9$. The renormalization constants which describe the mixing of evanescent operators into physical ones read $Z_{E_1,8g} = -4/3$, $Z_{E_2,8g} = 4/9$ and $Z_{E_3,8g} = -4/9$. Again, all IR poles cancel in $C_{8,e}^{g(1)}$ and the result coincides with the one obtained with the other method.

We now recall that the relevant quantity entering the calculation of BR_γ is not $C_7^\gamma(\mu_b)$ with $\mu_b = \mathrm{O}(m_b)$ but a combination $C_7^{\gamma,\rm eff}(\mu_b)$ of this Wilson coefficient and of the coefficients of the four quark operators. This combination is the coefficient of $\langle Q_7^\gamma \rangle^{(0)}$ calculated on-shell. It follows from this definition that, unlike C_7^γ, the effective coefficient is regularization scheme independent at LO [26] and does not depend on the basis of physical operators. In NDR the two combinations relevant for $B \to X_s\gamma$ and $B \to X_s g$ are [6]

$$
\begin{aligned}
C_7^{\gamma,\mathrm{eff}}(\mu) &= C_7^{\gamma}(\mu) + \sum_{i=1}^{10} y_i\, C_i(\mu)\,, \\
C_8^{g,\mathrm{eff}}(\mu) &= C_8^{g}(\mu) + \sum_{i=1}^{10} z_i\, C_i(\mu)\,,
\end{aligned}
\tag{2.17}
$$

where $y = (0, 0, -1/3, -4/9, -20/3, -80/9, 1/9, 4/27, 20/9, 80/27)$ and $z = (0, 0, 1, -1/6, 20, -10/3, -1/3, 1/18, -20/3, 10/9)$. The $\mathrm{O}(\alpha)$ contributions to the Wilson coefficients at $\mu = M_W$ in our operator basis eq. (2.2) can be found from those in the operator basis of [7] after a basis transformation which in four dimensions is simply

$$
\vec{C}(\mu) = \hat{\mathcal{R}}^T \vec{C}'(\mu)\,,
\tag{2.18}
$$

where $\vec{C}'$ are the Wilson coefficients in the basis of [7]. They are given in eqs. (8.111)–(8.117) of that review. The matrix $\hat{\mathcal{R}}$ is the extension of the same matrix of [29] and is needed only for the physical operators Q_1–Q_{10}, Q_7^γ and Q_8^g:

$$
\hat{\mathcal{R}} = \begin{pmatrix}
2 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{9} & -\frac{2}{3} & \frac{1}{36} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{4}{3} & 0 & -\frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{4}{9} & \frac{8}{3} & -\frac{1}{36} & -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -\frac{1}{8} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 4 & -\frac{1}{24} & -\frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{8} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & -1 & \frac{1}{24} & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\tag{2.19}
$$

In fact, beyond the leading order, the operator basis must be supplemented by a definition of the evanescent operators. This definition corresponds to a choice of scheme and it is different, for instance, in the *standard* basis of [7] and in the operator basis of [6, 29, 30]. On the other hand, a change of scheme can be in general accommodated by an additional non-linear term in the transformation eq. (2.18)

$$
\vec{C}(\mu) = \left(1 + \frac{\alpha_s(\mu)}{4\pi}\Delta\hat{r}_s^T + \frac{\alpha}{4\pi}\Delta\hat{r}_e^T\right)\hat{\mathcal{R}}^T\vec{C}'(\mu)\,,
\tag{2.20}
$$

where $\Delta\hat{r}_s$ and $\Delta\hat{r}_e$ are matrices that depend on the way the projection on the space of physical operators is implemented in the effective theory calculation (which in turn corresponds to a definition of evanescent operators). At the order we are interested in, they affect only C_1 and C_2.

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In the following, for definiteness, we follow the convention of [29, 30], whose basis of physical operators is a subset of eq. (2.2). Recalling that $C^{(1)}_{2,e}(M_W)$ was obtained in [2] in the *standard* basis, we have calculated the matrix $\Delta\hat{r}_e$ that connects the two different schemes. As a result, the non-vanishing $\mathrm{O}(\alpha)$ contributions to the Wilson coefficients of the four quark operators and of the evanescent operators at $\mu = M_W$ are given by

$$
\begin{aligned}
C^{(1)}_{2,e}(M_W) &= -\frac{22}{9} + \frac{4}{3}\ln\frac{M_Z^2}{M_W^2} + \frac{1}{9}\,,\\
C^{(1)}_{3,e}(M_W) &= -\frac{1}{s_W^2}\left(\frac{4}{9}B_0(x_t) + \frac{2}{9}C_0(x_t)\right),\\
C^{(1)}_{5,e}(M_W) &= \frac{1}{s_W^2}\left(\frac{1}{9}B_0(x_t) + \frac{1}{18}C_0(x_t)\right),\\
C^{(1)}_{7,e}(M_W) &= 4C_0(x_t) + \widetilde{D}_0(x_t) - \frac{1}{s_W^2}\left(\frac{10}{3}B_0(x_t) - \frac{4}{3}C_0(x_t)\right),\\
C^{(1)}_{9,e}(M_W) &= \frac{1}{s_W^2}\left(\frac{5}{6}B_0(x_t) - \frac{1}{3}C_0(x_t)\right),\\
C^{E(1)}_{1,e}(M_W) &= \frac{1}{s_W^2}\left(\frac{4}{3}B_0(x_t) + \frac{2}{3}C_0(x_t)\right),\\
C^{E(1)}_{2,e}(M_W) &= 4C_0(x_t) + \widetilde{D}_0(x_t) + \frac{1}{s_W^2}\left(10B_0(x_t) - 4C_0(x_t)\right),\\
C^{E(1)}_{3,e}(M_W) &= 4C_0(x_t) + \widetilde{D}_0(x_t)\,,
\end{aligned}
\tag{2.21}
$$

with

$$
\begin{aligned}
B_0(x_t) &= -\frac{x_t}{4(x_t-1)} + \frac{x_t}{4(x_t-1)^2}\ln x_t\,,\\
C_0(x_t) &= \frac{x_t(x_t-6)}{8(x_t-1)} + \frac{x_t(2+3x_t)}{8(x_t-1)^2}\ln x_t\,,\\
\widetilde{D}_0(x_t) &= \frac{16-48x_t+73x_t^2-35x_t^3}{36(x_t-1)^3} + \frac{-8+32x_t-54x_t^2+30x_t^3-3x_t^4}{18(x_t-1)^4}\ln x_t\,.
\end{aligned}
\tag{2.22}
$$

Here we have left explicit the extra scheme dependent term $1/9$ in $C^{(1)}_{2,e}(M_W)$: it is numerically very small.

The final results for $C^{\gamma(1)}_{7,e}$ and $C^{g(1)}_{8,e}$ are quite lengthy. We give instead two accurate approximate formulas for the $\mathrm{O}(g^2)$ contributions to $C_7^{\gamma,\mathrm{eff}}(M_W)$ and $C_8^{g,\mathrm{eff}}(M_W)$, which are valid when the effective hamiltonian is normalized in terms of G_F as in eq. (2.1):

$$
\begin{aligned}
C^{\gamma,\mathrm{eff}(1)}_{7,e}(\mu_W) = \frac{1}{s_W^2}\Bigg[&1.11 - 1.15\left(1-\frac{\overline{M}_t^2}{170^2}\right) - 0.444\ln\frac{M_H}{100} - 0.21\ln^2\frac{M_H}{100} -\\
&- 0.513\ln\frac{M_H}{100}\,\ln\frac{\overline{M}_t}{170}\Bigg] + \left(\frac{8}{9}C_7^{\gamma(0)} - \frac{104}{243}\right)\ln\frac{\mu_W^2}{M_W^2}\,,
\end{aligned}
$$

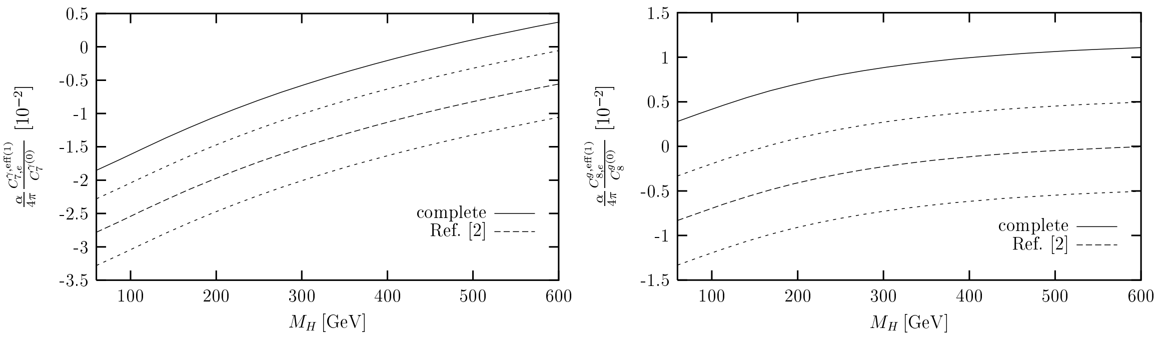

Figure 2: Electroweak corrections to the Wilson coefficients $C_7^\gamma(M_W)$ and $C_8^g(M_W)$. The dashed lines represent the results of [2] with their error estimates, the solid lines the complete corrections to the Wilson coefficients at M_W.

$$C_{8,e}^{g,\text{eff}(1)}(\mu_W) = \frac{1}{s_W^2}\left[-0.143 + 0.156\left(1 - \frac{\overline{M}_t^2}{170^2}\right) - 0.129\ln\frac{M_H}{100} - 0.0244\ln^2\frac{M_H}{100} - \right.$$
$$\left. - 0.037\ln\frac{M_H}{100}\,\ln\frac{\overline{M}_t}{170}\right] + \left(\frac{4}{9}C_8^{g(0)} - \frac{4}{3}C_7^{\gamma(0)} - \frac{58}{81}\right)\ln\frac{\mu_W^2}{M_W^2}\,. \quad (2.23)$$

Here M_H is the Higgs boson mass expressed, like $\overline{M}_t$ in GeV. In eq. (2.9) we use the coupling $\alpha(\mu_W) \approx 1/128$, while in general we employ $s_W^2 = 0.23$, corresponding to $g^2 = 4\sqrt{2}G_F M_W^2$, $M_W = 80.45\,\text{GeV}$ and $M_Z = 91.1875\,\text{GeV}$. Equations (2.23) reproduce accurately (within 1.5%) the analytic results in the ranges $80\,\text{GeV} < M_H < 300\,\text{GeV}$ and $160\,\text{GeV} < \overline{M}_t < 180\,\text{GeV}$. We stress that eqs. (2.23) are independent of the choice of the scale μ_t in the QCD top mass definition: it is sufficient to calculate $\overline{M}_t(\mu_t)$ and employ it in eqs. (2.23). Different choices of μ_t lead to different NLO QCD corrections, but they are higher order effects as far as the present calculation is concerned. The μ_W dependence of the effective coefficients agrees with [3, 5, 6].

The size of the electroweak corrections to $C_7^{\gamma,\text{eff}}$ and $C_8^{g,\text{eff}}$ relative to the one-loop results is shown in figure 2 as a function of the Higgs mass. To compare directly the results in eq. (2.23) with the approximate ones in [2], we have used the same central value $\overline{M}_t = 175.5\,\text{GeV}$ in the plots. First, notice that the Higgs mass dependence is identical, as should be expected since all the diagrams involving the Higgs boson also involve a charged boson. Therefore, these diagrams are not sensitive to the Z-W mass difference or to $\mathrm{O}(s_W^2)$ couplings. Numerically, we see from figure 2 that the difference is larger than estimated in [2]. Although an expansion of the results in powers of s_W^2 converges quickly, it turns out that its second term of $\mathrm{O}(s_W^2)$ is larger than naively expected, and that the two-loop correction is very sensitive to the M_Z-M_W difference.

3. QED-QCD evolution and the decay $B \to X_s\gamma$

The relevant quantity in the evaluation of BR_γ is the effective Wilson coefficient of the magnetic operator at a scale $\mu_b \approx m_b$. In the resummation of QED and QCD logarithms one usually keeps only terms linear in the electromagnetic coupling α, whose running is also neglected. In this case, the general structure of the evolution at $\mathrm{O}(\alpha\alpha_s^n L^n)$ is well known, see for instance [7, 8]. Using the notation $\vec{C}^T = (C_1, \ldots, C_{10}, C_7^{\gamma,\mathrm{eff}}, C_8^{g,\mathrm{eff}})$ and restricting to the physical (on-shell) operators in eq. (2.2), the coefficients at a scale μ are given in terms of the coefficients at the scale M_W by

$$\vec{C}(\mu) = \vec{C}^{(0)}(\mu) + \frac{\alpha_s(\mu)}{4\pi}\vec{C}_s^{(1)}(\mu) + \frac{\alpha}{4\pi}\vec{C}_e^{(1)}(\mu) = \hat{U}(\mu, M_W, \alpha)\,\vec{C}(M_W)\,, \tag{3.1}$$

where

$$\hat{U}(\mu, M_W, \alpha) = \hat{U}^{(0)}(\mu, M_W) + \hat{U}^{(1)}(\mu, M_W) + \frac{\alpha}{4\pi}\left[\hat{R}^{(0)}(\mu, M_W) + \hat{R}^{(1)}(\mu, M_W)\right]. \tag{3.2}$$

The first two terms give the pure QCD evolution. The matrices $\hat{U}^{(i)}$ and $\hat{R}^{(i)}$ are determined by the anomalous dimension matrix of the operators in question and by the QCD β function. Explicit expressions for $\hat{U}^{(0)}$, $\hat{U}^{(1)}$, and $\hat{R}^{(0)}$ can be extracted from [6, 23]. $\hat{R}^{(1)}$ is presently unknown: it requires the evaluation of the two and three-loop anomalous dimension matrix at $\mathrm{O}(\alpha\alpha_s)$. From the point of view of the expansion in α_s in the renormalization group improved perturbation theory, $\hat{U}^{(0)}$ and $\hat{U}^{(1)}$ are $\mathrm{O}(1)$ and $\mathrm{O}(\alpha_s)$, respectively. $\hat{R}^{(0)}$ and $\hat{R}^{(1)}$ are $\mathrm{O}(1/\alpha_s)$ and $\mathrm{O}(1)$, respectively. Expanding in α_s and α we obtain

$$\begin{aligned}\vec{C}(\mu) = {}& \hat{U}^{(0)}(\mu, M_W)\left[\vec{C}^{(0)} + \frac{\alpha_s(M_W)}{4\pi}\vec{C}_s^{(1)}\right] + \hat{U}^{(1)}(\mu, M_W)\vec{C}^{(0)} + \\ & + \frac{\alpha}{4\pi}\left[\hat{U}^{(0)}(\mu, M_W)\vec{C}_e^{(1)} + \hat{R}^{(0)}(\mu, M_W)\left(\vec{C}^{(0)} + \frac{\alpha_s(M_W)}{4\pi}\vec{C}_s^{(1)}\right) + \right. \\ & \left. + \hat{R}^{(1)}(\mu, M_W)\vec{C}^{(0)}\right].\end{aligned} \tag{3.3}$$

All the coefficients on the right-hand side are understood at the scale M_W. The first line results from pure QCD evolution. The second and third line mixes QED, electroweak, and QCD effects. After the calculation of the two missing elements of $\vec{C}_e^{(1)}$ in section 2, the only unknown part of eq. (3.3) relevant for (chromo)magnetic decays is the last term (third line).

Turning to the particular case of $C_7^{\gamma,\mathrm{eff}}$ and neglecting the unknown term $\hat{R}^{(1)}(\mu, M_W)\vec{C}^{(0)}$, we see from eq. (3.3) and [6, 7] that the $\mathrm{O}(\alpha)$ terms are given by

$$C_{7,e}^{\gamma,\mathrm{eff}(1)}(\mu_b) = C_{7,e}^{\gamma,U}(\mu_b) + C_{7,e}^{\gamma,R}(\mu_b)\,. \tag{3.4}$$

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The first term on the right-hand side corresponds to $\hat{U}^{(0)}(\mu, M_W)\vec{C}_e^{(1)}$ in eq. (3.3) and takes the form

$$\begin{aligned} C_{7,e}^{\gamma,U}(\mu_b) = {} & \eta^{16/23} C_{7,e}^{\gamma,\mathrm{eff}(1)}(M_W) + \frac{8}{3}\left(\eta^{14/23} - \eta^{16/23}\right) C_{8,e}^{g,\mathrm{eff}(1)}(M_W) - \\ & - (0.448 - 0.49\,\eta)\, C_{2,e}^{(1)}(M_W) + (0.362 - 0.454\,\eta)\, C_{3,e}^{(1)}(M_W) + \\ & + (5.57 - 5.86\,\eta)\, C_{5,e}^{(1)}(M_W) + (0.321 - 0.47\,\eta)\, C_{7,e}^{(1)}(M_W) + \\ & + (1.588 - 2.89\,\eta)\, C_{9,e}^{(1)}(M_W)\,, \end{aligned} \tag{3.5}$$

where $\eta = \alpha_s(M_W)/\alpha_s(\mu_b) \approx 0.56$ for $\mu_b = m_b$. Here we have given analytically only the cofactors of $C_7^{\gamma,\mathrm{eff}}$ and $C_8^{g,\mathrm{eff}}$. The other terms are more involved and are given in an approximate form, valid within 1% for values of η between 0.5 and 0.6. However, they can all be easily determined from the anomalous dimension matrices given in [6].[5] The last four terms have been given in [2] in the operator basis of [7].

The second term in eq. (3.4) corresponds to $\hat{R}^{(0)}(\mu, M_W)(\vec{C}^{(0)} + \alpha_s(M_W)/(4\pi)\vec{C}_s^{(1)})$ in eq. (3.3) and is given by

$$\begin{aligned} C_{7,e}^{\gamma,R}(\mu_b) = {} & \frac{4\pi}{\alpha_s(\mu_b)} \Bigg[\left(\frac{88}{575}\eta^{16/23} - \frac{40}{69}\eta^{-7/23} + \frac{32}{75}\eta^{-9/23} \right) \times \\ & \times \left(C_7^{\gamma,\mathrm{eff}(0)}(M_W) + \frac{\alpha_s(M_W)}{4\pi} C_{7,s}^{\gamma,\mathrm{eff}(1)}(M_W) \right) + \\ & + \left(\frac{640}{1449}\eta^{14/23} - \frac{704}{1725}\eta^{16/23} + \frac{32}{1449}\eta^{-7/23} - \frac{32}{575}\eta^{-9/23} \right) \times \\ & \times \left(C_8^{g,\mathrm{eff}(0)}(M_W) + \frac{\alpha_s(M_W)}{4\pi} C_{8,s}^{g,\mathrm{eff}(1)}(M_W) \right) - \\ & - 0.0449 + 0.2504\eta - 0.236\eta^2 \Bigg] + \\ & + (0.15 - 0.178\,\eta)\,\eta\, C_{1,s}^{(1)}(M_W) - (0.381 - 0.556\,\eta)\,\eta\, C_{4,s}^{(1)}(M_W)\,. \end{aligned} \tag{3.6}$$

The $\mathrm{O}(\alpha_s)$ coefficients $C_{i,s}^{(1)}(M_W)$ can be found in [10, 28, 30]. The approximate expressions are valid within 1% for $0.5 < \eta < 0.6$.

We are now ready to give a numerical value for the $\mathrm{O}(\alpha)$ Wilson coefficient at $\mu_b = 4.7\,\mathrm{GeV}$ using eq. (3.6). Renormalizing the top mass at $\mu_t = \overline{M}_t = 165\,\mathrm{GeV}$, we find

$$C_{7,e}^{\gamma,\mathrm{eff}(1)}(\mu_b) = 4.172 - 1.312 \ln\frac{M_H}{100} - 0.615 \ln^2\frac{M_H}{100} + 2.360\,, \tag{3.7}$$

where the first three terms correspond to $C_{7,e}^{\gamma,U}$ and the last one to $C_{7,e}^{\gamma,R}$. Notice that for a light Higgs boson the first term (formally $\mathrm{O}(\alpha\alpha_s^n L^n)$) is twice the second one (formally $\mathrm{O}(\alpha\alpha_s^{n-1} L^n)$). We interpret this as evidence that purely electroweak $\mathrm{O}(\alpha_W)$ effects are dominant with respect to purely QED effects.

[5] Table 2 in [2] allows to change from the basis of [6] to that of [7].

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To see how electroweak corrections affect the calculation of BR_γ it is sufficient to recall that, for $\mu_b = m_b$, the perturbative QCD expression for the $b \to s\gamma(g)$ decay is proportional to

$$\left| C_7^{\gamma,\mathrm{eff}}(m_b) + \vec{C}^{(0)}(m_b) \cdot \left(\frac{\alpha_s(m_b)}{4\pi} \vec{r}_s + \frac{\alpha}{4\pi} \vec{r}_e \right) \right|^2 + B(E_0) \,, \tag{3.8}$$

where E_0 is the maximal photon energy in the b-quark frame and $B(E_0)$ originates from bremsstrahlung diagrams. $\vec{r}_s$ and $\vec{r}_e$ originate from the $\mathrm{O}(\alpha_s)$ and $\mathrm{O}(\alpha)$ matrix elements of the physical operators. $\vec{r}_s$ has been computed in [31] with the exception of $r_{3,s}, \ldots, r_{10,s}$. It is easy to see from these papers that[6]

$$r_{1,e} = -\frac{2}{9} r_{2,s} \,, \qquad r_{2,e} = -\frac{1}{6} r_{2,s} \,, \qquad r_{7\gamma,e} = \frac{1}{12} r_{7\gamma,s} - \frac{1}{4} r_{8g,s} \,, \tag{3.9}$$

while $r_{8g,e} = 0$. Numerically, the effect of $r_{2,s}$ and $r_{7\gamma,s}$ in the calculation of the inclusive branching ratio is quite important. On the other hand, the $\mathrm{O}(\alpha_s)$ contribution to $B(E_0)$ changes BR_γ by less than 4% if $1\,\mathrm{GeV} < E_0 < 2\,\mathrm{GeV}$. We therefore conclude that the only potentially relevant QED matrix elements are the virtual corrections parameterized by $r_{1,e}$, $r_{2,e}$ and $r_{7\gamma,e}$. Hence, in our numerics we will neglect the unknown last term in eq. (3.3), the unknown QCD matrix elements (which are in any case suppressed by small Wilson coefficients), and the remaining real and virtual QED contributions to the matrix elements. It has been observed in [1] that splitting the charm and top quark contributions in eq. (3.8) and normalizing them in an asymmetric way leads to an improved perturbative QCD expansion. In the evaluation of the electroweak corrections, however, this would be an unnecessary complication.

We stress that, as we neglect the last term in eq. (3.3) and some contributions to the matrix elements, our evaluation of $\mathrm{O}(\alpha\alpha_s^n L^n)$ effects in $B \to X_s\gamma$ is incomplete, although we are confident that it should provide a good approximation. Our numerical result is valid in the NDR scheme supplemented by the definition of evanescent operators of [29, 30]. An analysis of the way the scheme dependent terms recombine can be found in [8]. The scheme dependence of our result is introduced in eq. (2.21) and in the $r_{i,e}$. The one from eq. (2.21) is numerically negligible (less than 0.01% on BR_γ). All the residual scheme dependent pieces would be cancelled by corresponding terms in the anomalous dimension matrix at $\mathrm{O}(\alpha\alpha_s)$, if it were available.

Additional $\mathrm{O}(\alpha)$ contributions are introduced by normalizing BR_γ in terms of the semileptonic branching ratio, as there are well-known QED corrections to the semileptonic decay amplitude [32]. Unfortunately, only the leading logarithmic term is known. The final $\mathrm{O}(\alpha)$ contribution to the expression under absolute value in

[6] Because of the definition of $B(E_0)$ adopted in [1], we use here $r_{7\gamma,s} = \frac{8}{9}(4-\pi^2)$ as given in [31]. This convention is different from the one of [30].

eq. (3.8) is therefore [1]

$$\varepsilon_{\rm ew} = \frac{\alpha}{4\pi}\left[C_{7,e}^{\gamma,{\rm eff}(1)}(m_b) + r_{1,e}C_1^{(0)}(m_b) + r_{2,e}C_2^{(0)}(m_b) + \right.$$
$$\left. + r_{7\gamma,e}C_7^{\gamma,{\rm eff}(0)}(m_b) - 4\, C_7^{\gamma,{\rm eff}(0)}(m_b)\ln\frac{M_Z}{m_b}\right]. \tag{3.10}$$

Using the reference value $M_H = 115\,{\rm GeV}$ and $m_c/m_b = 0.22$ as in [1], we find numerically

$$\varepsilon_{\rm ew} = 0.0025 + 0.0014 + 0.0004 + 0.0028 = 0.0071\,, \tag{3.11}$$

which updates eq. (4.6) of [1]. Here the first and second terms correspond to the U and R components of $C_{7,e}^{\gamma,{\rm eff}(1)}(\mu_b)$ (in [1] they were 0.0035 and 0.0012, respectively). The third term derives from the QED matrix elements and was not included in the paper mentioned above. The last term, 0.0028, is due to the QED corrections to the semileptonic decay amplitude and is the same as in [1]. Notice that the first term, although formally suppressed with respect to the second one, is larger, as it incorporates all purely electroweak contributions. The total effect of the QED and electroweak corrections in $\varepsilon_{\rm ew}$ on the branching ratio is a 3.6% reduction while the $O(\alpha\alpha_s^n L^n)$ contributions alone lead to a 1.6% reduction. As different contributions accidentally compensate each other, $\varepsilon_{\rm ew}$ is almost exactly the same that was used in [1]. Incorporating all perturbative and non-perturbative QCD corrections and using the same numerical inputs as in [1], we therefore obtain for different values of the cutoff photon energy in the $\bar{B}$ meson frame the following results, very close to those given in that paper:

$$\begin{aligned} {\rm BR}\left[\bar{B}\longrightarrow X_s\gamma\right]_{E_\gamma > m_b/20} &= (3.74 \pm 0.30)\times 10^{-4}\,, \\ {\rm BR}\left[\bar{B}\longrightarrow X_s\gamma\right]_{E_\gamma > 1.6\,{\rm GeV}} &= (3.61 \pm 0.30)\times 10^{-4}\,. \end{aligned} \tag{3.12}$$

Here the errors are estimates of theoretical errors also based on the analysis of [1]. One can compare the first of these two results with the present experimental world average ${\rm BR}_\gamma = (3.23 \pm 0.42)\times 10^{-4}$ [33].

4. Conclusions

We have calculated the complete $O(\alpha)$ Wilson coefficients relevant for radiative weak decays and described the implementation of $O(\alpha\alpha_s^n L^n)$ effects in detail, including also the dominant QED matrix elements. The final impact of these contributions on the branching ratio of $B \to X_s\gamma$ is roughly 1.6% for a light Higgs mass $M_H \approx 100\,{\rm GeV}$, and decreases slowly for larger values of M_H.

We have discussed in detail the role played by unphysical operators in the calculation. We have adopted two different methods to regulate the IR divergences and clarified the subtleties that arise in the two cases. In contrast to the off-shell $\mathrm{O}(\alpha_s)$ calculation [10, 11], evanescent operators turn out to play a crucial role in the $\mathrm{O}(\alpha)$ computation. We have also explained the relevance of gauge variant operators in our calculation.

Our results improve upon existent calculations [2, 3, 5, 6] and put electroweak corrections to $B \to X_s\gamma$ on a firmer basis, although numerically the change is negligible. The dependence of $C_7^{\gamma,\mathrm{eff}}(\mu_b)$ on heavy degrees of freedom is now completely known at $\mathrm{O}(\alpha)$. We have also included the dominant $\mathrm{O}(\alpha)$ matrix elements. Still, not all the $\mathrm{O}(\alpha\alpha_s^n L^n)$ contributions to radiative decays are under control. The uncalculated corrections are related to the QED-QCD evolution (last term in eq. (3.3)) and to some suppressed QED matrix elements. As the electroweak contributions to QCD and electroweak penguin operators are relevant to our discussion, some two-loop QCD matrix elements are also still missing. The incompleteness of our calculation makes it scheme-dependent, but, as we have noted above, the scheme dependence is remarkably small. On the other hand, the calculation of the missing contributions would require a significant effort. In the meanwhile, we note that: (i) the leading $\mathrm{O}(\alpha\alpha_s^{n-1}L^n)$ corrections affect the branching ratio of $B \to X_s\gamma$ only in a minor way (-0.6%) and (ii) the QED matrix elements are very small, although formally of the same order of the matching corrections. Therefore, one might expect the missing subleading QED effects to be eventually small.

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